COMPARISONS OF IMPROVED BONFERRONI AND SIDAK/SLEPIAN BOUNDS WITH APPLICATIONS TO NORMAL MARKOV PROCESSES

BY

DONALD R. HOOVER

TECHNICAL REPORT NO. 412
FEBRUARY 23, 1989

PREPARED UNDER CONTRACT
N00014-86-K-0156 (NR-042-267)
FOR THE OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

COMPARISONS OF IMPROVED BONFERRONI AND SIDAK/SLEPIAN BOUNDS WITH APPLICATIONS TO NORMAL MARKOV PROCESSES

BY

DONALD R. HOOVER

TECHNICAL REPORT NO. 412
FEBRUARY 23, 1989

Prepared Under Contract
N00014-86-K-0156 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

ABSTRACT

The recent literature contains theorems improving on both the Standard Bonferroni inequality (Hoover (1988-a)) and the Sidak/Slepian inequalities (Glaz and Johnson (1984)). The application of these improved theorems to upper bounds for non coverage of simultaneous confidence intervals on multivariate normal variables is explored. The improved Bonferroni upper bounds will always apply, while improved Sidak/Slepian bounds only apply to special cases. The improved Sidak/Slepian upper bound, if it applies, is always superior to the equivalent improved Bonferroni bound. This improvement, however, is not great when both methods are used to determine upper bounds for Type I error in the range of .01 to .10. It is shown that improved Sidak/Slepian bounds will apply to Normal Markov Processes, a commonly occurring and easily identifiable class of multivariate normal variables.

Key Words Include:

GLAZ AND JOHNSON BOUND, HUNTER BOUND, IMPROVED BONFERRONI BOUND, MRR_2, MTP_2, NORMAL MARKOV PROCESS, SIDAK BOUND, SLEPIAN INEQUALITY, SIMULTANEOUS INFERENCE

1. INTRODUCTION

Recently, there has been considerable interest in improving on the Bonferroni and Sidak/Slepian upper bounds for simultaneous coverage of multivariate normal confidence intervals (Hunter (1976), Glaz and Johnson (1984), Hoover (1988-a)). Let $\underset{\sim}{X} = (X_1,\ldots,X_n)$ be random variables with a joint multivariate normal distribution. Define C_i as some constant greater than zero for i=1,...,n. Often C_i will be $\sigma_i \cdot C$ where σ_i^2 is the variance of X_i and $C > 0$. Let E_i i=1,...,n be events defined as follows: CASE I - (Two sided symmetric confidence interval): E_i is the event $|X_i| \notin (-C_i, C_i)$. CASE II - (one sided upper limit interval): E_i is the event $X_i \notin (-\infty, C_i)$. CASE III (one sided lower limit interval): E_i is the event $X_i \notin (-C_i, \infty)$.

Let E_i^c be the complement of event E_i and $E = \underset{i=1}{\overset{n}{\cup}} [E_i]$. Finally, define $P_i = P\{E_i\}$, $P_i^c = P\{E_i^c\} = (1-P_i)$ and $P_E = P\{E\}$.

Usually, due to limitations on numerical integration, it will be impossible to calculate P_E exactly. Therefore, in order to be conservative, it is desired to calculate U where U is an upper bound for P_E. The closer U is to P_E, the better the conservative bound is. The importance of having good upper bounds U arises from the many applications of simultaneous confidence intervals (See Khatri (1967); Barlow and Proschan (1975); and Bauer and Hackel (1985)). Sometimes, it may be of interest to obtain L, a lower bound for P_E.

The Bonferroni and Sidak/Slepian upper bounds (U_B and U_S respectively) and the Slepian lower bound (L_S) are now presented. Let $\underset{\sim}{X}$, E_i, E, P_i and P_E be as defined above. Let ρ_{ij} be the correlation of X_i and X_j $1 \le i < j \le n$; then

(1) Bonferroni (1936) Inequality (Holds for all types of confidence intervals)

$$P_E \le U_B = \sum_{i=1}^{n} P_i \ .$$

(2) Sidak (1968) Inequality (Holds for Case I two sided confidence intervals)

$$P_E \le U_S = 1 - \prod_{i=1}^{n} P_i^c \ .$$

(3) Slepian (1962) Upper and Lower Bound Inequalities (Holds for CASE II and CASE III one sided confidence intervals)

$$\text{If } \rho_{ij} \geq 0 \text{ for all } i,j \text{ then } P_E \leq U_S = 1 - \prod_{i=1}^{n} P_i^c .$$

$$\text{If } \rho_{ij} \leq 0 \text{ for all } i,j \text{ then } P_E \geq L_S = 1 - \prod_{i=1}^{n} P_i^c .$$

Note that the Slepian upper and Sidak bounds are identical except that the Sidak bound holds for two sided confidence intervals while the Slepian upper bound holds for one sided intervals.

It has been proven, (Dunn (1958)), that U_S in (2) and (3) is less than (better than) the corresponding U_B. It is also clear that U_S is non degenerate, $U_S \leq 1$ while U_B can be degenerate, $U_B > 1$ is possible.

Neither U_B nor U_S uses the intersection structure within (E_1, E_2, ..., E_n). That is, U_B and U_S each produce the same values for n independent events as they do for n positively correlated events. This is true even though P_E can be much lower for n positively correlated events than it is for n independent events. For instance, suppose $n = 6$, and $P_i = 0.5$ for $i = 1,\ldots,6$. If $E_1,\ldots,E_6$ are jointly independent, then $P_E = 1 - \prod_{i=1}^{6} (P_i^c) = .2649$; while under the most extreme correlation: $\left(E_1 = E_2 = \ldots = E_6\right)$, $P_E = P_1 = 0.5$. However, in both of the cases described above, the bound U_B is $6(.05) = .30$ and the bound U_S, if it applies, is $1 - (.95)^6 = .2649$.

In the above example and in general, the bounds U_B and U_S approximate P_E well when the E_i are independent, but do more poorly when the X_i are correlated. A similar problem holds for L_S. Two methods that can incorporate interaction (collinearity) between subsets of size k of the E_i to produce sharper bounds for P_E have recently been proposed. The first method, presented in Section 2, is an extension of the Bonferroni theorem while the second method, presented in Section 3, extends the Sidak/Slepian results.

2. THE EXTENDED BONFERRONI METHOD

Theorem 2.1 The Extended Bonferroni Bound (Hunter (1976) k=2, Hoover (1988-a) k>2)

Let X_i, E_i, E and P_E be as defined in the first paragraph of section 1. Let k be an integer: $1 \leq k \leq n$. Suppose that it is possible to integrate over k or

fewer dimensions or otherwise determine $P\left\{\bigcup_{i=j-k+1}^{j-1} E_i\right\}$ and

$P\left\{\bigcup_{i=j-k+1}^{j} E_i\right\}$ for $j=k,\ldots,n$.

Then

$$(4) \quad P_E \leq U_{B,k} = P\left\{\bigcup_{i=1}^{k} E_i\right\} + \sum_{j=k+1}^{n}\left(P\left\{\bigcup_{i=j-k+1}^{j} E_i\right\} - P\left\{\bigcup_{i=j-k+1}^{j-1} E_i\right\}\right)$$

The proof follows from Theorem 1 from Hoover (1988-a) with the set S_j containing $\{j-1, j-2, \ldots, j-k+1\}$.

This theorem, like the standard Bonferroni bound, is distribution free. The events $E_1, E_2, \ldots, E_n$ can be any events (not merely those defined by confidence intervals on multivariate normal variables) and the inequality (4) will hold. When the E_i are as defined in Section 1, it is currently computationally possible to integrate over multivariate normal densities of up to 4 dimensions, (see Schervish (1984)), and thus implement this theorem for $k \leq 4$. Undoubtedly as computers and computational methods improve, it will become possible to implement this theorem for even larger values of k.

When k=1, Theorem 2.1 reduces to the Standard Bonferroni Bound. It is shown in Hoover that $U_{B,k}$ always is monotonically decreasing (improving) as k increases. However, the extended Bonferroni upper bound, like the Bonferroni upper bound, can be degenerate (See Table 6.1).

3. THE EXTENDED SIDAK/SLEPIAN METHOD

The extended Sidak and Slepian bounds will now be presented. These bounds will hold only for those normal (and other) distributions which have certain MTP_2 or MRR_2 properties. Definitions 3.1, 3.2 and 3.3 are given to define these MTP_2 and MRR_2 properties.

Definition 3.1 (Karlin and Rinot (1980-a)) A real function (or joint density) of two variables f(W,Y) is Totally Positive of Order Two, $\left(TP_2\right)$ if for all fixed values $w_1<w_2$ and $y_1<y_2$,

$$(5) \qquad f(w_1,y_1)\, f(w_2,y_2) - f(w_1,y_2)\, f(w_2,y_1) \geq 0$$

Definition 3.2 (Karlin and Rinot (1980-a)) A real valued function (or joint density) of n variables $f(X_1,\ldots,X_n)$ is Multivariate Totally Positive of Order Two, $\left(MTP_2\right)$, if for any pair of variables X_i and X_j, the function $f(X_i, X_j)$ with the remaining variables fixed is TP_2.

Definition 3.3 (Karlin and Rinot (1980-b)). The bivariate function of Definition 3.1 is Reverse Rule of Order Two (RR_2) and the multivariate function of Definition 3.2 is Multivariate Reverse Rule of Order Two (MRR_2) if the direction of the inequality in (5) is reversed.

A pair of variables (X_1, X_2) is said to be TP_2 (RR_2) if its joint density is TP_2 (RR_2) respectively. A sequence of variables $(X_1, \ldots, X_n)$ is said to be MTP_2 (MRR_2) if its joint density is MTP_2 (MRR_2) respectively.

Theorem 3.1 The Extended Sidak/Slepian Bounds (Glaz and Johnson (1984))

Let X_i, E_i, E, and P_E be as defined in the first paragraph of section 1, with the exception that $\underset{\sim}{X}$ is now allowed to have any distribution not only the multivariate normal. Let k be an integer, $1 \leq k \leq n$. Assume that it is possible to integrate over k dimensions or otherwise determine $P\left\{E_j^c \middle| \bigcap_{i=j-k+1}^{j-1} E_i^c\right\}$. Then

(a) (Extended Slepian Upper Bound) If $\underset{\sim}{X}$ is MTP_2 it follows when the events E_i are either all one sided upper limit intervals or all one sided lower limit intervals that

$$P_E \leq U_{S,k} = 1-P\left\{\bigcap_{i=1}^{k} E_i^c\right\} \cdot \prod_{j=k+1}^{n} P\left\{E_j^c \middle| \bigcap_{i=j-k+1}^{j-1} E_i^c\right\}$$

(b) (Extended Slepian Lower Bound) If $\underset{\sim}{X}$ is MRR_2 it follows when the events E_i are either all one sided upper limit intervals or all one sided lower limit intervals that

$$P_E \geq L_{S,k} = 1-P\left\{\bigcap_{i=1}^{k} E_i^c\right\} \cdot \prod_{j=k+1}^{n} P\left\{E_j^c \middle| \bigcap_{i=j-k+1}^{j-1} E_i^c\right\}$$

(c) (Extended Sidak Bound) If $|\underset{\sim}{X}|$ is MTP_2 it follows when the events E_i are all two sided symmetric confidence intervals that

$$P_E \leq U_{S,k} = 1-P\left\{\bigcap_{i=1}^{k} E_i^c\right\} \cdot \prod_{j=k+1}^{n} P\left\{E_j^c \middle| \bigcap_{i=j-k+1}^{j-1} E_i^c\right\}$$

The proofs of statements (a), (b) and (c) follow from Theorems 2.3 and 2.8 in Glaz and Johnson (1984) when one realizes that:

(i) $P\{X_i \in (-C_i, C_i)\} = P\{|X_i| \in (-\infty, C_i)\}$ (an infinite interval)

(ii) If γ_k is a $\binom{\text{lower}}{\text{upper}}$ bound for $P\left\{\bigcap_{i=1}^{n} E_i^c\right\}$ then $1-\gamma_k$ is a $\binom{\text{upper}}{\text{lower}}$ bound for $P_E = P\left\{\bigcup_{i=1}^{n} E_i\right\}$.

Note that when $\underset{\sim}{X}$ is normal, $|\underset{\sim}{X}|$ will never be MRR_2 and thus there is no "Sidak like" lower bound for P_E in the two sided case.

For k=1, the bound in Theorem 3.1 is equivalent to the Sidak/Slepian Bounds. $U_{S,k}$ like U_S is non degenerate and is monotonically decreasing with k, while $L_{S,k}$ is monotonically increasing with k; see Glaz and Johnson (1984). As mentioned earlier, the extended Sidak and Slepian bounds do not hold for all normal distributions. Section 5 describes in detail some important distributions for which they do hold. Finally, Theorem 3.1 can be applied to multivariate t distributions; see Glaz and Johnson (1984).

4. EXTENDED SIDAK/SLEPIAN BOUND IS SUPERIOR TO EXTENDED BONFERRONI BOUND

Two methods have been proposed for obtaining upper bounds which utilize interactions between subsets of events, namely the extended Bonferroni ($U_{B,k}$) and the extended Sidak/Slepian ($U_{S,k}$) bounds. It is therefore of interest to know which, if either, gives better results. The next theorem shows that whenever the extended Sidak/Slepian method applies, it produces better upper bounds than does the equivalent extended Bonferroni Method.

Theorem 4.1. Superiority of $U_{S,k}$ over $U_{B,k}$, Glaz (1988), Hoover (1988-b)
Let $\underset{\sim}{X}$, E_i, P_E, $U_{B,k}$ and $U_{S,k}$ be as defined in Theorem 3.1. Let Σ be the covariance of $\underset{\sim}{X}$. Also assume that $\underset{\sim}{X}$ (or $|\underset{\sim}{X}|$) and E_i have the required properties for Theorem 3.1 to apply. Then for any fixed k:

(i) $U_{S,k} = U_{B,k} = P_E$ for k=n

(ii) $U_{S,k} \leq U_{B,k}$ for k<n

(iii) If $\underset{\sim}{X}$ has a multivariate normal (or t) distribution and the covariance (Σ) is full rank, then $U_{S,k} < U_{B,k}$ for $k < n$.

Proof - This Theorem has been independently proven by Glaz (1988) and Hoover (1988-b). The proof from Hoover (1988-b) is in the appendix.

Not only is $U_{S,k}$ superior to $U_{B,k}$ for any fixed value of $k < n$, but also $U_{S,k}$ is computationally no more difficult to calculate than is $U_{B,k}$. This is because $P\left\{\bigcap_{i=j-k+1}^{j} E_i^c\right\} = 1 - P\left\{\bigcup_{i=j-k+1}^{j} E_i\right\}$. Therefore, whenever the

distributional conditions are met for using the extended Sidak/Slepian upper bound, this bound should be used instead of the corresponding extended Bonferroni upper bound.

5. SOME IMPORTANT DISTRIBUTIONS TO WHICH EXTENDED SIDAK/SLEPIAN BOUNDS APPLY

Since it has been proven that whenever the extended Sidak/Slepian upper bound is applicable, it gives better results than does the comparable extended Bonferroni bound and since an extended Slepian lower bound exists, it is of interest to identify important situations in which the extended Sidak/Slepian bounds apply.

The extended Slepian upper (lower) bounds require that the variables $(\underset{\sim}{X})$ have an MTP_2 (MRR_2) structure. It follows from Bolviken (1982) and Karlin and Rinott (1980-a), that if $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \Sigma)$ then $\underset{\sim}{X}$ is MTP_2 iff $\tilde{\rho}_{ij} \geq 0$ for $i \neq j$ where $\tilde{\rho}_{ij}$ is the partial correlation of X_i and X_j given all other variables in the vector $\underset{\sim}{X}$. Karlin and Rinot (1980-b) give sufficient conditions on Σ for a multivariate normal vector $\underset{\sim}{X}$ to be MRR_2.

The extended Sidak bound requires $|\underset{\sim}{X}|$ to be MTP_2. It follows from Karlin and Rinot (1980-a) and Bolviken (1982) that if $\underset{\sim}{X}$ is a multivariate normal vector, then $|\underset{\sim}{X}|$ is MTP_2 iff there exists some diagonal matrix D with $|D_{ii}| = 1$, $i=1,\ldots, n$, such that $\underset{\sim}{Y} = D\,\underset{\sim}{X}$ and $\tilde{\rho}_{ij}^{\,y} \geq 0$ for all i,j where $\tilde{\rho}_{ij}^{\,y}$ is the partial correlation of Y_i and Y_j given all other variables in the vector $\underset{\sim}{Y}$.

It turns out that many multivariate normal processes of interest have the above properties. The next theorem shows that multivariate vectors for which each successive component is a successive time realization of a Markov process have MTP_2 and/or MRR_2 properties.

Theorem 5.1 MTP_2 and MRR_2 Properties of Normal Markov Processes.
Let $\underset{\sim}{X} = (X_1,\ldots,X_n)$ be a Normal Markov Process, that is: $X_1, X_2,\ldots, X_n$ are successive outcomes of a Markov Process for $i=1,\ldots,n$ and $(X_1,\ldots X_n)$ has a multivariate normal distribution. Then

(a) $|\underset{\sim}{X}|$ is MTP_2

(b) if $\rho_{i,i+1} \geq 0$ for $i=1,\ldots,n-1$ then $\underset{\sim}{X}$ is MTP_2.

(c) if $\rho_{i,i+1} \leq 0$ for $i=1,\ldots,n-1$ then $\underset{\sim}{X}$ is MRR_2.

Proof

(a) Since $|X_{i+1}| = |-X_{i+1}|$ and at least one of the following {Corr(X_i, X_{i+1}), Corr (X_i, $-X_{i+1}$)} is non negative for $i=1,\ldots,n-1$, it follows from Bolviken (1982) that ($|X_i|$, $|X_{i+1}|$) is TP_2. Since (X_1, $X_2,\ldots,X_n$) is a Markov

chain, ($|X_1|$, $|X_2|$,..., $|X_n|$) is also a Markov chain. Therefore, ($|X_1|$, $|X_2|$,..., $|X_n|$) is a Markov chain with TP_2 transition probabilities and by proposition 3.10 in Karlin and Rinot (1980-a) it follows that $|\underset{\sim}{X}|$ = ($|X_1|$, $|X_2|$,..., $|X_n|$) is MTP_2.

(b) Since Corr (X_i,X_{i+1}) is non negative, it follows from Karlin and Rinot (1980-a) that (X_i,X_{i+1}) is TP_2. Therefore, X_1,X_2,...,X_n is a Markov chain with TP_2 transition probabilities and by proposition 3.10 in Karlin and Rinot (1980-a) it follows that (X_1,X_2,...,X_n) is MTP_2.

(c) Since Corr (X_i, X_{i+1}) is nonpositive, (X_i, - X_{i+1}) is TP_2 which means that (X_i, X_{i+1}) is RR_2. Therefore, X_1, X_2,..., X_n is a Markov chain with RR_2 transition probabilities; so by a slight modification of proposition 3.10 in Karlin and Rinot (1980-a), it follows that (X_1, X_2, ..., X_n) is MRR_2.

There are many normal Markov Processes which are of interest in practice. Some important examples are given in Section 6. The extended Bonferroni upper bound has previously been applied and/or recommended for these examples for which a superior extended Sidak/Slepian upper bound could be applied. In Section 6, it will be shown how much improvement the extended Sidak upper bound gives over the equivalent extended Bonferroni upper bound for these examples. Extended Slepian bounds are not considered since in practice one sided confidence intervals are not of as much interest as are two sided confidence intervals.

6. IMPROVEMENTS USING $U_{S,k}$ GIVES OVER USING $U_{B,k}$ FOR SELECTED NORMAL MARKOV PROCESSES

The extended Sidak bound is compared to the equivalent extended Bonferroni bound for the following three examples of normal Markov processes.

6.1 AR(1) TIME SERIES

Consider (X_1,...,X_n) as n successive observations of a stationary time series with $X_i \sim N(0,\sigma^2)$ i=1,...,n, and σ^2 known. Also assume $X_{i+1} = \rho \cdot X_i + \varepsilon_i$ for i=1,...,n-1 where $0 \leq \rho \leq 1$ and $\varepsilon_i \sim N(0, \sigma^2(1-\rho^2))$. Then $\underset{\sim}{X}$ = (X_1,...,X_n) is an AR(1) Markov process as described in Box and Jenkins (1976) and $\underset{\sim}{X}$ has a multivariate normal distribution with $Var(X_i)=\sigma^2$ and $Cov(X_i,X_{i+1}) = \rho\sigma^2$. By Theorem 5.1, $|\underset{\sim}{X}|$ is MTP_2 and therefore, by Theorem 4.1. the extended Sidak method applies to two sided confidence intervals involving $\underset{\sim}{X}$. Since the covariance of $\underset{\sim}{X}$ is full rank, by Theorem 4.1, the extended Sidak/Slepian upper bound is better than the extended Bonferroni bound. Table 6.1 compares $U_{S,2}$ and $U_{S,3}$ with $U_{B,2}$ and $U_{B,3}$ on AR(1) processes using selected values of n, ρ and C.

TABLE 6.1 *

Extended Sidak and Extended Bonferroni Upper Bounds for Non Simultaneous Coverage of Two Sided Confidence Intervals on AR(1) Time Series Variables where $C_i = C.\sigma$ for i=1,...,n.

C = 1.96

		k=2		k=3	
ρ	n	$U_{B,2}$	$U_{S,2}$	$U_{B,3}$	$U_{S,3}$
.3	5	0.23047	0.21801	0.22381	0.21781
	10	0.45606	0.38689	0.43831	0.38646
	50	> 1	0.91244	> 1	0.91207
	100	> 1	0.99231	> 1	0.99225
.7	5	0.18368	0.17689	0.17553	0.17248
	10	0.35104	0.31194	0.32900	0.30207
	50	> 1	0.83596	> 1	0.82132
	100	> 1	0.97627	> 1	0.96746
.9	5	0.13141	0.12883	0.12220	0.12125
	10	0.23317	0.21823	0.20863	0.19996
	50	> 1	0.67125	0.90005	0.62237
	100	> 1	0.88867	> 1	0.85225
.99	5	0.07631	0.07604	0.07155	0.07147
	10	0.10920	0.10759	0.09652	0.09579
	50	0.37235	0.32415	0.29625	0.26872
	100	0.70129	0.52252	0.54590	0.43915

C = 3.00

		k=2		k=3	
ρ	n	$U_{B,2}$	$U_{S,2}$	$U_{B,3}$	$U_{S,3}$
.3	5	0.01331	0.01327	0.01326	0.01324
	10	0.02657	0.02632	0.02644	0.02624
	50	0.13265	0.12469	0.13187	0.12430
	100	0.26526	0.23381	0.26365	0.23310
.7	5	0.01166	0.01163	0.01148	0.01146
	10	0.02286	0.02268	0.02385	0.02225
	50	0.11245	0.10674	0.10961	0.10440
	100	0.22444	0.20172	0.21864	0.19745
.9	5	0.00861	0.00860	0.00820	0.00819
	10	0.01601	0.01593	0.01489	0.01484
	50	0.07517	0.07265	0.06846	0.06471
	100	0.14912	0.13897	0.13543	0.12722
.99	5	0.00469	0.00469	0.00438	0.00437
	10	0.00717	0.00716	0.00631	0.00631
	50	0.02704	0.02676	0.02189	0.02172
	100	0.05188	0.05070	0.04137	0.04066

* Numerical Integration to a relative accuracy of 0.00001 done using Shervish's (1984) program

6.2 SEQUENTIAL TESTING USING MEANS OF NORMAL VARIABLES

A

Let X_i be iid $N(\mu, \sigma^2)$ with σ^2 known. Suppose it is of interest to test H_o: $\mu=0$ and also that it is expensive or inconvenient to sample the X_i. Then one approach for testing H_o involves a sequential sampling of the X_i with at each step calculating $Y_j = \frac{1}{\sigma \cdot j} \sum_{i=1}^{j} X_i$ for $j=1,\ldots,n$. Also, at each step, a two sided confidence interval for μ of the form $Y_j - C\cdot\sqrt{Var(Y_j)} \leq \mu \leq Y_j + C\cdot\sqrt{Var(Y_j)}$ where $C>0$ is constructed. H_o is rejected immediately and sampling is stopped at the first step j where the confidence interval does not include zero. If all confidence intervals include zero, then H_o is accepted.

Bauer and Hackel (1985) suggest using $U_{B,2}$ as an upper bound for the type I error because of the high correlation between Y_j and Y_{j+1} which equals $\left(\frac{j}{j+1}\right)^{1/2}$. But since $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ is a normal Markov process with a full rank covariance, then by Theorems 5.1 and 4.1, $U_{S,2}$ will give even better upper bounds for type I error. Table 6.2 compares $U_{B,2}$ and $U_{S,2}$ for various values of n and C.

TABLE 6.2*

Extended Sidak and Extended Bonferroni Upper Bounds for Type One Error of Sequential Testing Using Means of Normal Variables (See Section 5.2)

n	C=1.96		C=2.50		C=3.00	
	$U_{B,2}$	$U_{S,2}$	$U_{B,2}$	$U_{S,2}$	$U_{B,2}$	$U_{S,2}$
10	0.23785	0.22229	0.06724	0.06592	0.01616	0.01608
20	0.35794	0.31520	0.10302	0.09921	0.02513	0.02489
50	0.59846	0.46895	0.17493	0.16255	0.04325	0.04245
80	0.77248	0.55809	0.22705	0.20563	0.05640	0.05500
100	0.87067	0.60159	0.25647	0.22896	0.06283	0.06201

* Numerical Integration to a relative accuracy of 0.00001 done using Schervish's (1984) Program.

6.3 DETECTION OF A CHANGE POINT IN A NORMAL SERIES

Under H_o, Let $X_1,\ldots,X_n$ be independently distributed $N(\mu,\sigma^2)$ with σ^2 known. While under H_A, let the n observations be independent realizations of the series $X_i\ldots,X_{m-1}$ and $X_m,\ldots, X_n$ distributed as $N(\mu_1,\sigma^2)$ and $N(\mu_2, \sigma^2)$ respectively. The "Change Point" m is unknown. To test H_o vs. H_A, confidence intervals of the form $(W_j - C(var(W_j)), W_j + C(var(W_j))$ are constructed about the standardized likelihood ratio statistic

$$W_j = \frac{\sum_{i=1}^{j} X_i/j - \sum_{i=j+1}^{n} X_i/(n-j)}{\sigma\left(\frac{1}{j} + \frac{1}{n-j}\right)^{1/2}} \quad \text{for } j=1,\ldots,n-1 .$$

H_o is rejected if and only if one of the above confidence intervals does not cover zero. Both Bauer and Hackel (1985) and Talwar (1983) suggest using $U_{B,2}$ as an upper bound for the type I error because of the high correlation between W_j and W_{j+1} which equals $\left(\frac{j(n-j-1)}{(j+1)(n-j)}\right)^{1/2}$. Since $W_1, W_2, \ldots, W_n$ are realizations of a normal Markov chain with a full rank covariance, then by Theorem 4.1, $U_{S,2}$ will give a better upper bound to the type I error than will $U_{B,2}$. Table 6.3 compares $U_{S,2}$ with $U_{B,2}$ for various values of C and n.

TABLE 6.3*

Extended Sidak and Extended Bonferroni Upper Bounds for Type One Error of Testing

for a Change Point in a Normal Series

n	C=1.96		C=2.50		C=3.00	
	$U_{B,2}$	$U_{S,2}$	$U_{B,2}$	$U_{S,2}$	$U_{B,2}$	$U_{S,2}$
10	0.30629	0.27531	0.08348	0.08129	0.01954	0.01941
20	0.49298	0.40797	0.13985	0.13240	0.03358	0.03135
50	0.87262	0.60395	0.25327	0.22669	0.06212	0.06042
80	> 1	0.70358	0.33538	0.28845	0.08284	0.07975
100	> 1	0.74822	0.38170	0.32108	0.09454	0.09048

* Numerical integration to a relative accuracy of 0.00001 done using Schervish's (1984) program

6.4 DISCUSSION

Define $U_{S,k}/U_{B,k}$ as the efficiency ratio (ER) of the extended Bonferroni upper bound relative to the extended Sidak upper bound. By Theorem 4.1, ER $\leq$ 1. The closer this ratio is to one, the better the extended Bonferroni does with respect to the extended Sidak.

In all of the tables, the efficiency ratio is decreasing in n and increasing in C. For instance, in Table 5.2, the ER for n=10 and C=1.96 is 0.9345 while the ER for n=100 and C=1.96 is 0.6908, and the ER for n=10 and C=3.00 is 0.9950. The lowering of ER with respect to n is a result of the inductive nature shown in step 2 of the proof of Theorem 4.1. In this step, it was shown that $[U_{B,k} - U_{S,k}]$ for j+1 events was larger than $[U_{B,k} - U_{S,k}]$ for

the first j of these events. The increase in ER as C becomes larger is probably related to a similar phenomenon observed with the original Bonferroni and Sidak bounds. As P_i becomes small, which happens when C becomes large, $U_B/U_S \to 1$ (see Dunn (1958)).

In Table 6.1, $\rho_{i,i+1}$ is equal to a constant value ρ for all i=1,2,...,n-1. The ER decreases in ρ. For instance, with C=1.96, n=10 and k=2; the ER is 0.8483 when ρ=0.3 and 0.9852 when ρ=0.99. Thus, the extended Sidak improves on the extended Bonferroni more when the correlation is lower between the normal variables involved in the adjacent events.

Table 6.1 also has values of $U_{B,3}$ and $U_{S,3}$ in addition to values of $U_{B,2}$ and $U_{S,2}$. When C is small (C = 1.96), $U_{B,3}$ and $U_{B,2}$ are often degenerate and/or even $U_{B,3}$ is larger than $U_{S,2}$. For instance, when C = 1.96, ρ=0.7 and n=50, both $U_{B,2}$ and $U_{B,3}$ are greater than one. Meanwhile, $U_{S,2}$, which must be non-degenerate, is 0.82132 which, of course, is less than $U_{B,3}$.

Most applied statisticians are interested in upper bounds for the probability of type I error which are either 0.01, 0.05 or 0.10. The ER's observed in these tables when upper bounds were between 0.01 and 0.10 ranged from 0.8307 to $\simeq$ 1.0. In many of these cases, the extended Sidak bounds were not drastically lower than were the corresponding extended Bonferroni bounds.

Something which should be considered when using multivariate normal approximations is that the density being approximated may not be MTP_2 even though the normal approximation is MTP_2. In this situation, extended Sidak bounds may be less appropriate than are extended Bonferroni bounds. Since $U_{B,k}$ and $U_{S,k}$ are close together when they are in the range of interest to applied statisticians, the investigator does not lose much by using the more conservative and possibly more correct extended Bonferroni bound.

REFERENCES

Armitage, P.; McPherson, C. and Rowe, B. (1969), Repeated Significance Tests on Accumulating Data, JRSS Ser A., 132, 235-244

Barlow, R E. and Proschan, F. (1975), "Statistical Theory of Reliability and Lire Testing," New York: Holt, Rinehart and Winston

Bauer, P. and Hackel, P. (1985), The Application of Hunters Inequality in Simultaneous Testing, Biom J., 27, No 1, 25-38

Bolviken, E. (1982), Probability Inequalities for the Multivariate Normal with Non-negative Partial Correlations, Scand J Stat 9 47-58

Bonferroni, C. (1936), Teoria Statistica Classie Calcolo Delle Probabililta, Pubbl. R. 1st. Sugges. Sci. Econ. Comm. Firenze, 8 1-62

Box, G. and Jenkins, G. (1976) "Time Series Analysis Forecasting and Control," San Francisco: Holden Day

Dunn, O. J., (1958) Estimation of the Means of Dependent Variables, Ann Math Statist, 29, 1095-1111

Glaz, J. (1988), A Comparison of Bonferroni-Type and Product Type Inequalities in the Presence of Dependence (Submitted for publication in the Proceedings from the 1987 Symposium on Statistics and Probability, Hidden Valley, PA. (to be published by IMS.))

Glaz, J. and Johnson, B. (1984), Probability Inequalities for Multivariate Distributions with Dependence Structure, JASA 79 No. 380, 436-440

Hoover, D. (1988-a), "Subset Complement Addition Upper Bounds, An Improved Inclusion-Exclusion Method" (Submitted for Publication and also presented at the 1987 ASA Annual meeting.)

Hoover, D. (1988-b), "Improved Bonferroni and Sidak/Slepian Bounds: Comparisons and Applications." University of South Carolina Statistics Technical Report No. 134.

Hunter, D. (1976), An Upper Bound for the Probability of a Union, J Appl Prob 13, 597-603

Karlin, S. and Rinot, Y. (1980-a), Classes of Orderings of Measures and Related Correlation Inequalities. 1. Multivariate Totally Positive Distributions., J of Multivariate Analysis, 10, 467-498

Karlin, S. and Rinot, Y. (1980-b), Classes of Orderings of Measures and Related Correlation Inequalities. 2. Multivariate Reverse Rule Distributions., J of Multivariate Analysis, 10, 499-516.

Khatri, C. G. (1967), On Certain Inequalities for Normal Distributions and Their Applications to Simultaneous Confidence Bounds, Annals of Mathematical Statistics, 38, 1853-1867.

Schervish, M. (1984) Multivariate Normal Probabilities with Error Bound, Applied Statistics, 33, 81-94

Sidak, Z. (1967) Rectangular Confidence Regions for Means of Multivariate Normal Distributions, JASA, 66, 626-633

Slepian, D., (1962) The One Sided Barrier Problem for Gaussian Noise, Bell System Tech J, 41, 463-501

Stoline, M. (1983), The Hunter Method of Simultaneous Inference and its Recommended Use for Applications Having Large Known Correlation Structures. JASA, 78, 366-370

Talwar, P. (1983), Detecting a Shift Location, J of Econometrics, 23, 353-367

APPENDIX

Proof of Theorem 4.1

STEP 1 For k=n, $U_{S,k} = 1-P\left\{\bigcap_{i=1}^{n} E_i^c\right\} = P_E = P\left\{\bigcup_{i=1}^{n} E_i\right\} = U_{B,k}$

and hence statement (i) of Theorem 4.1 has been proven.

STEP 2 For k<n (By induction)

Let $(\underset{\sim}{X}_j)$ be the first j elements of the vector $\underset{\sim}{X}$ for j=k, k+1, ...,n. It is a fact that if $\underset{\sim}{X}$ is MTP_2, then $\underset{\sim}{X}_j$ is also MTP_2. (See Karlin and Rinot (1980)). So define E(j) to be $\bigcup_{i=1}^{j} E_i$ and P(j) to be P{E(j)} where E_i is as defined in Section 1. Let $U_{B,k}^{(j)}$ and $U_{S,k}^{(j)}$ be the extended Bonferroni and extended Sidak/Slepian upper bounds for P(j). It will now be shown that if $U_{S,k}^{(j)} \le U_{B,k}^{(j)}$ then $U_{S,k}^{(j+1)} \le U_{B,k}^{(j+1)}$.

$$U_{S,k}^{(j+1)} = 1 - P\left\{ E_{j+1}^c \,\middle|\, \bigcap_{i=j-k+2}^{j} E_i^c \right\} P\left\{ \bigcap_{i=1}^{k} E_i^c \right\} \prod_{i=k+1}^{j} P\left\{ E_i^c \,\middle|\, \bigcap_{h=j-k+1}^{i-1} E_h^c \right\}$$

$$= 1 - P\left\{ E_{j+1}^c \,\middle|\, \bigcap_{i=j-k+2}^{j} E_i^c \right\} (1 - U_{S,k}^{(j)})$$

$$= 1 - \left(1 - U_{S,k}^{(j)}\right) P\left\{ \bigcap_{i=j-k+2}^{j+1} E_i^c \right\} + P\left\{ \bigcap_{i=j-k+2}^{j} E_i^c \right\}$$

Since $U_{S,K}^{(j)} \le U_{B,k}^{(j)}$ by assumption

$$\le 1 - \left(1 - U_{B,k}^{(j)}\right) P\left\{ \bigcap_{i=j-k+2}^{j+1} E_i^c \right\} + P\left\{ \bigcap_{i=j-k+2}^{j} E_i^c \right\}$$

Since $P\left\{\left[\bigcup_{i=j-k+2}^{j} E_i\right]^c\right\} = P\left\{\bigcap_{i=j-k+2}^{j} E_i^c\right\}$

$$= 1 - \left(1 - U_{B,k}^{(j)}\right) P\left\{E_{j+1}^{c} \cap \left[\bigcup_{i=j-k+2}^{j} E_i\right]^{c}\right\} + P\left\{\bigcap_{i=j-k+2}^{j} E_i^{c}\right\}$$

$$= 1 - (1 - U_{B,k}^{(j)}) \frac{P\left\{\left[\bigcup_{i=j-k+2}^{j} E_i\right]^{c}\right\} - P\left\{E_{j+1} \cap \left[\bigcup_{i=j-k+2}^{j} E_i\right]^{c}\right\}}{P\left\{\left[\bigcup_{i=j-k+2}^{j} E_i\right]^{c}\right\}}$$

$$= 1 - (1 - U_{B,k}^{(j)}) + \frac{1-U_{B,k}^{(j)}}{P\left\{\bigcap_{i=j-k+2}^{j} E_i^{c}\right\}} P\left\{E_{j+1} \cap \left[\bigcup_{i=j-k+2}^{j} E_i\right]^{c}\right\}$$

(*) Since $\dfrac{1-U_{B,k}^{(j)}}{P\left\{\bigcap_{i=j-k+2}^{j} E_i^{c}\right\}} \leq \dfrac{P\left\{\bigcap_{i=1}^{j} E_i^{c}\right\}}{P\left\{\bigcap_{i=j-k+2}^{j} E_i\right\}^{c}} \leq 1$

$$(5) \qquad \leq U_{B,k}^{(j)} + P\left\{E_{j+1} \cap \left[\bigcup_{i=j-k+2}^{j} E_i\right]^{c}\right\}$$

$$= U_{B,k}^{(j)} + P\left\{\bigcup_{i=j-k+2}^{j+1} E_i\right\} - P\left\{\bigcup_{i=j-k+2}^{j} E_i\right\}$$

expanding $U_{B,k}^{(j)}$

$$= P\left\{\bigcup_{i-1}^{k} E_i\right\} + \sum_{i=k+1}^{j+1} \left[P\left\{\bigcup_{h=i-k+1}^{i} E_h\right\} - P\left\{\bigcup_{h=i-k+1}^{i-1} E_h\right\}\right]$$

$$= U_{B,k}^{(j+1)}$$

But now the proof is finished since in Step 1 it was shown that $U_{S,k}^{(k)} = U_{B,k}^{(k)}$ which implies by the inequality of Step 2 that $U_{S,k}^{(k+1)} \leq U_{B,k}^{(k+1)}$ and by induction on this inequality n-k times that $U_{S,k} \overset{\text{def}}{=} U_{S,k}^{(n)} \leq U_{B,k}^{(n)} \overset{\text{def}}{=} U_{B,k}$.

and hence statement (ii) of Theorem 4.1 is proven.

STEP 3. When the E_i are confidence intervals for multivariate normal (or t) variables with a full rank covariance matrix, then the inequalities (*) and (5) are strict and thus statement (iii) of Theorem 4.1 follows.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 412	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Comparisons Of Improved Bonferroni And Sidak/ Slepian Bounds With Applications To Normal Markov Processes		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Donald R. Hoover		8. CONTRACT OR GRANT NUMBER(s) N00014-86-K-0156
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics & Probability Program Code 1111		12. REPORT DATE February 23, 1989 13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Glaz and Johnson Bound, Hunter Bound, Improved Bonferroni Bound, MRR_2, MTP_2, Normal Markov Process, Sidak Bound, Slepian Inequality, Simultaneous Inference

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

PLEASE SEE FOLLOWING PAGE.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

TECHNICAL REPORT NO. 412

20. ABSTRACT

The recent literature contains theorems improving on both the Standard Bonferroni inequality (Hoover (1988-a)) and the Sidak/Slepian inequalities (Glaz and Johnson (1984)). The application of these improved theorems to upper bounds for non coverage of simultaneous confidence intervals on multivariate normal variables is explored. The improved Bonferroni upper bounds will always apply, while improved Sidak/Slepian bounds only apply to special cases. The improved Sidak/Slepian upper bound, if it applies, is always superior to the equivalent improved Bonferroni bound. This improvement, however, is not great when both methods are used to determine upper bounds for Type I error in the range of .01 to .10. It is shown that improved Sidak/Slepian bounds will apply to Normal Markov Processes, a commonly occurring and easily identifiable class of multivariate normal variables.